A Mathematical Formulation of the Equivalence Principle

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Abstract —**A mathematical formulation of the equivalence principle is presented. This may lead to a better understanding and easier applications of the principle.**

I. Introduction

THE EQUIVALENCE principle in electromagnetics has been well known for a long time, having been presented by Harrington [1] in a descriptive manner in his book. Recently, this principle has found many applications in problems involving the interaction of EM fields with material bodies. In these applications, accurate mathematical formulations of this principle are needed. The purpose of this paper is to present a mathematical formulation of the equivalence principle that may lead to a better understanding of the principle and make its application easier.

II. Mathematical Formulation

Consider a problem with a geometry as depicted in Fig. 1. This geometry consists of region 2 with complex permittivity and permeability (ϵ_2, μ_2), the volume V_2, the boundary surface S, and the electric and magnetic source currents $(\vec{J}_2, \vec{M}_2)$ within V_2. Region 2 is surrounded by region 1 of infinite volume V_1 that has electric parameters of (ϵ_1, μ_1) and source currents of $(\vec{J}_1, \vec{M}_1)$ within V_1.

We aim to find the EM fields in regions 1 and 2 in terms of the given source currents and equivalent surface currents on S. In the process, we will derive a mathematical formulation of the well-known equivalence principle.

Maxwell's equations for regions 1 and 2 are

$$\begin{cases} \nabla \times \vec{E}_1 = -\vec{M}_1 - j\omega\mu_1\vec{H}_1 \\ \nabla \times \vec{H}_1 = \vec{J}_1 + j\omega\epsilon_1\vec{E}_1 \end{cases} \quad \text{in } V_1 \tag{1}$$

$$\begin{cases} \nabla \times \vec{E}_2 = -\vec{M}_2 - j\omega\mu_2\vec{H}_2 \\ \nabla \times \vec{H}_2 = \vec{J}_2 + j\omega\epsilon_2\vec{E}_2 \end{cases} \quad \text{in } V_2. \tag{2}$$

Let us consider region 1 first and apply the vector Green's

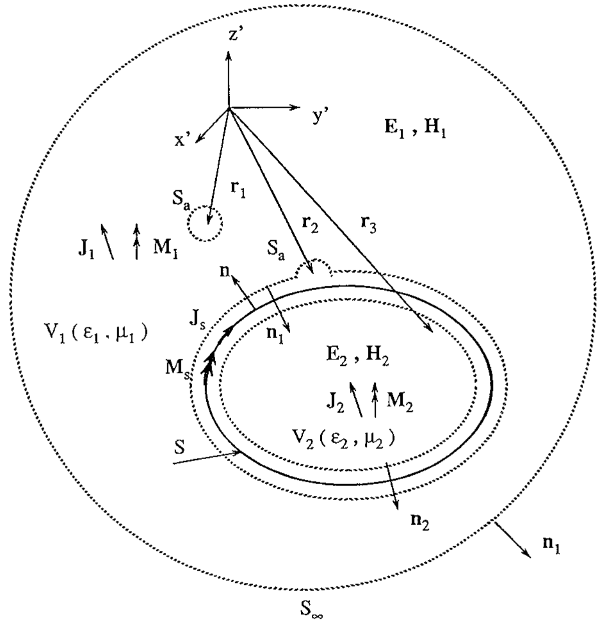

Fig. 1. Geometry of the problem: region 2 with volume V_2, boundary surface S, electric parameters (ϵ_2, μ_2), and source currents (J_2, M_2) is surrounded by region 1 with infinite volume V_1, electric parameters (ϵ_1, μ_1), and source currents (J_1, M_1). (E_1, H_1) constitute the EM field in V_1 and (E_2, H_2) that in V_2.

theorem to V_1:

$$\int_{V_1} \left(\vec{Q}\cdot\nabla'\times\nabla'\times\vec{P} - \vec{P}\cdot\nabla'\times\nabla'\times\vec{Q}\right) dv' = \int_{S_1} \left(\vec{Q}\times\nabla'\times\vec{P} - \vec{P}\times\nabla'\times\vec{Q}\right)\cdot \overrightarrow{ds} \tag{3}$$

where $\vec{Q}$ and $\vec{P}$ are two vector functions which are continuous up to their second derivatives within V_1. S_1 is the total boundary surface for V_1. We choose

$$\vec{P}(\vec{r}\,') = \vec{E}_1(\vec{r}\,') \tag{4}$$

and

$$\vec{Q}(\vec{r}\,') = \hat{a}\phi_1(\vec{r}\,', \vec{r}) = \hat{a}\exp\left(-j\beta_1|\vec{r}\,'-\vec{r}|\right)/|\vec{r}\,'-\vec{r}| \tag{5}$$

where

$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1}\,.$$

In the above equations, $\vec{r}\,'$ is an arbitrary source (integrating) point and $\vec{r}$ is a designated field (observation) point, $\vec{E}_1(\vec{r}\,')$ is the electric field at $\vec{r}\,'$ within V_1, $\hat{a}$ is a constant unit vector, and ϕ_1 is the unbounded Green's function for

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region 1. It is noted that if $\vec{r}$ is within V_1, $\vec{Q}_1$ will not be continuous at $\vec{r}\,'=\vec{r}$ and it is necessary to remove this singularly point before (3) can be applied.

When the field point $\vec{r}$ is an interior point within V_1, such as $\vec{r}_1$ in Fig. 1, $\phi_1 \to \infty$ as $\vec{r}\,' \to \vec{r}_1$; thus we need to exclude this point with a small sphere having a small surface of S_a as depicted in Fig. 1. Then the total boundary surface S_1 for V_1 will consist of

$$S_1 = S + S_a + S_\infty$$

where S_∞ is the infinite spherical surface enclosing the outside of V_1.

The substitution of (4) and (5) into (3), with the help of (1) and after a lengthy manipulation [2], will lead to the following equation:

$$\int\left[-j\omega\mu_1\vec{J}_1\phi_1 - \vec{M}_1\times\nabla'\phi_1 + \frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' = \int_{S+S_a+S_\infty}\left[-j\omega\mu_1\left(\hat{n}_1\times\vec{H}_1\right)\phi_1 + \left(\hat{n}_1\times\vec{E}_1\right)\times\nabla'\phi_1 + \left(\hat{n}_1\cdot\vec{E}_1\right)\nabla'\phi_1\right]ds' \tag{6}$$

where ρ_1 is the electric source charge associated with $\vec{J}_1$ by the continuity equation of $\nabla\cdot\vec{J}_1 + j\omega\rho_1 = 0$.

It can be shown that

$$\int_{S_a}[\quad]\,ds' = 4\pi\vec{E}_1(\vec{r}_1)$$

$$\int_{S_\infty}[\quad]\,ds' = 0$$

based on the radiation condition. Thus, (6) becomes

$$\vec{E}_1(\vec{r}_1) = \frac{1}{4\pi}\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1 - \vec{M}_1\times\nabla'\phi_1 + \frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' - \frac{1}{4\pi}\int_S\left[-j\omega\mu_1\left(\hat{n}_1\times\vec{H}_1\right)\phi_1 + \left(\hat{n}_1\times\vec{E}_1\right)\times\nabla'\phi_1 + \left(\hat{n}_1\cdot\vec{E}_1\right)\nabla'\phi_1\right]ds'. \tag{7}$$

At this point, we can define the equivalent electric and magnetic surface currents as

$$\vec{J}_s \equiv \hat{n}\times\vec{H}_1 = -\hat{n}_1\times\vec{H}_1 \tag{8}$$

$$\vec{M}_s \equiv -\hat{n}\times\vec{E}_1 = \hat{n}_1\times\vec{E}_1 \tag{9}$$

where $\hat{n}$ is the unit vector pointing outward from region 2 on S, and $\hat{n}_1$ is the outgoing unit vector of region 1 on S.

Since the tangential components of $\vec{E}$ and $\vec{H}$ fields are continuous across S, $\vec{J}_s$ and $\vec{M}_s$ can also be expressed as

$$\vec{J}_s = \hat{n}\times\vec{H}_2 = \hat{n}_2\times\vec{H}_2 \tag{10}$$

$$\vec{M}_s = -\hat{n}\times\vec{E}_2 = -\hat{n}_2\times\vec{E}_2 \tag{11}$$

where $\hat{n}_2$ is the outgoing unit vector of region 2 on S, and it is in the same direction as $\hat{n}$.

We can also drive from (1) that

$$\hat{n}_1\cdot\vec{E}_1 = \frac{j}{\omega\epsilon_1}\nabla\cdot\left(\hat{n}_1\times\vec{H}_1\right) \tag{12}$$

if S is a smooth surface and no source current $\vec{J}_1$ is present at S. If we use (8), (12) can be rewritten as

$$\hat{n}_1\cdot\vec{E}_1 = \frac{-j}{\omega\epsilon_1}\nabla\cdot\left(\vec{J}_s\right) = \frac{-1}{\epsilon_1}\rho_s \tag{13}$$

where ρ_s is the equivalent electric surface charge associated with $\vec{J}_s$ by the continuity equation of $\nabla\cdot\vec{J}_s + j\omega\rho_s = 0$.

Substituting (8), (9), and (13) into (7) leads to

$$\vec{E}_1(\vec{r}_1) = \frac{1}{4\pi}\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1 - \vec{M}_1\times\nabla'\phi_1 + \frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' + \frac{1}{4\pi}\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1 - \vec{M}_s\times\nabla'\phi_1 + \frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds'. \tag{14}$$

The physical meaning of (14) is as follows: The electric field at an interior point $\vec{r}_1$ within V_1, $\vec{E}_1(\vec{r}_1)$, is maintained by the given source currents $(\vec{J}_1, \vec{M}_1)$ in V_1 and equivalent surface currents $(\vec{J}_s, \vec{M}_s)$ on the surface S while the medium of region 2 is replaced by that of region 1 and the source currents $(\vec{J}_2, \vec{M}_2)$ in V_2 are removed. This is because the parameters (ϵ_2, μ_2) and $(\vec{J}_2, \vec{M}_2)$ do not appear in (14) and the unbounded Green's function ϕ_1 appears in both the volume and surface integrals in (14). From the appearance of (14), $\vec{E}_1(\vec{r}_1)$ is maintained by the source currents $(\vec{J}_1, \vec{M}_1)$ and the equivalent surface currents $(\vec{J}_s, \vec{M}_s)$ located in the unbounded homogeneous region with electric parameters of (ϵ_1, μ_1).

Next, let us consider the case when the field point $\vec{r}$ is on the surface S, such as $\vec{r}_2$ in Fig. 1. For this case, we need to exclude the singularly point $\vec{r}_2$ from V_1 with a hemisphere which has a hemispherical surface S_a as shown in Fig. 1 before we can use (3). With this S_a, the surface integral over S_a in (6) becomes

$$\int_{S_a}[\quad]\,ds' = 2\pi\vec{E}_1(\vec{r}_2). \tag{15}$$

Therefore, (6) can be rearranged to give $\vec{E}_1(\vec{r}_2)$ as

$$\vec{E}_1(\vec{r}_2) = \frac{1}{2\pi}\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1 - \vec{M}_1\times\nabla'\phi_1 + \frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' + \frac{1}{2\pi}\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1 - \vec{M}_s\times\nabla'\phi_1 + \frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds'. \tag{16}$$

Comparing (16) with (14), there is a factor of 2 between them. The surface integral in (16) is a principal value integral which excludes the contribution from the singularity point.

Lastly, if the field point $\vec{r}$ is located outside V_1, or inside V_2, such as $\vec{r}_3$ in Fig. 1, ϕ_1 is continuous throughout V_1. Therefore, we do not need to create a small sphere to exclude the field point $\vec{r}_3$ from V_1. Thus, (6) becomes

$$\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1-\vec{M}_1\times\nabla'\phi_1+\frac{\rho_1}{\epsilon_1}\nabla'\phi\right]dv' = \int_{S+S_\infty}\left[-j\omega\mu\left(\hat{n}_1\times\vec{H}_1\right)\phi_1+\left(\hat{n}_1\times\vec{E}_1\right)\times\nabla'\phi_1 + \left(\hat{n}_1\cdot\vec{E}_1\right)\nabla'\phi_1\right]ds' \quad \text{for } \vec{r}=\vec{r}_3. \tag{17}$$

Since the surface integral over S_∞ is zero due to the radiation condition, (17) leads to

$$\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1-\vec{M}_1\times\nabla'\phi_1+\frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' = -\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1-\vec{M}_s\times\nabla'\phi_1+\frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds' \quad \text{for } \vec{r}=\vec{r}_3. \tag{18}$$

Now, if we try to express the electric field at $\vec{r}_3$ maintained by the given source currents $(\vec{J}_1, \vec{M}_1)$ in V_1 and the equivalent surface currents $(\vec{J}_s, \vec{M}_s)$ on S while replacing the medium in region 2 with that of region 1 and removing the source currents $(\vec{J}_2, \vec{M}_2)$ in V_2, we should have an expression for $\vec{E}_2(\vec{r}_3)$ of the following form:

$$\vec{E}_2(\vec{r}_3)=\frac{1}{4\pi}\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1-\vec{M}_1\times\nabla'\phi_1+\frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' + \frac{1}{4\pi}\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1-\vec{M}_s\times\nabla'\phi_1+\frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds' \quad \text{with } \vec{r}=\vec{r}_3. \tag{19}$$

Combining (18) and (19), we have

$$\vec{E}_2(\vec{r}_3)=0. \tag{20}$$

This is an interesting result. It means that if the source currents $(\vec{J}_2, \vec{M}_2)$ in V_2 are removed and the medium of region 2 is replaced by that of region 1 (to make the whole space homogeneous), then the source currents $(\vec{J}_1, \vec{M}_1)$ in V_1 and the equivalent surface currents $(\vec{J}_s, \vec{M}_s)$ on S will maintain a zero electric field at any point within region 2.

We can derive similar results for the $\vec{H}$ field in regions 1 and 2 in terms of $(\vec{J}_1, \vec{M}_1)$ and $(\vec{J}_s, \vec{M}_s)$:

$$\vec{H}_1(\vec{r}_1)=\frac{1}{4\pi}\int_{V_1}\left[-j\omega\epsilon_1\vec{M}_1\phi_1+\vec{J}_1\times\nabla'\phi_1+\frac{\rho_{m1}}{\mu_1}\nabla'\phi_1\right]dv' + \frac{1}{4\pi}\int_S\left[-j\omega\epsilon_1\vec{M}_s\phi_1+\vec{J}_s\times\nabla'\phi_1+\frac{\rho_{ms}}{\mu_1}\nabla'\phi_1\right]ds' \quad \text{for } \vec{r}=\vec{r}_1 \text{ (interior point within } V_1) \tag{21}$$

$$\vec{H}_1(\vec{r}_2)=\frac{1}{2\pi}\int_{V_1}\left[-j\omega\epsilon_1\vec{M}_1\phi_1+\vec{J}_1\times\nabla'\phi_1+\frac{\rho_{m1}}{\mu_1}\nabla'\phi_1\right]dv' + \frac{1}{2\pi}\int_S\left[-j\omega\epsilon_1\vec{M}_s\phi_1+\vec{J}_s\times\nabla'\phi_1+\frac{\rho_{ms}}{\mu_1}\nabla'\phi_1\right]ds' \quad \text{for } \vec{r}=\vec{r}_2 \text{ (surface point on } S) \tag{22}$$

$$\vec{H}_2(\vec{r}_3)=0 \quad \text{for } \vec{r}=\vec{r}_3 \text{ (outside of } V_1) \tag{23}$$

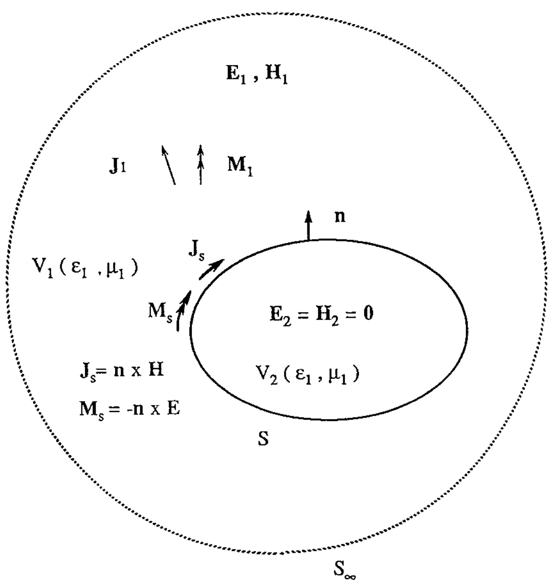

Fig. 2. When the source currents (J_2, M_2) in V_2 are removed and the medium of region 2 is replaced with that of region 1, the source currents (J_1, M_1) in V_1 and the equivalent surface currents (J_s, M_s) on S will maintain the correct EM field (E_1, H_1) in V_1 and zero EM field $(E_2 = H_2 = 0)$ in V_2.

where

$$\rho_{m1}=\frac{j}{\omega}\nabla\cdot\vec{M}_1 \quad \text{and} \quad \rho_{ms}=\frac{j}{\omega}\nabla\cdot\vec{M}_s.$$

The results obtained so far are consistent with the equivalence principle. The situation is depicted in Fig. 2.

We can repeat a similar derivation for region 2. Choosing

$$\vec{P}(\vec{r}\,')=\vec{E}_2(\vec{r}\,') \tag{24}$$

and

$$\vec{Q}(\vec{r}\,')=\hat{a}\phi_2(\vec{r}\,',\vec{r})=\hat{a}\exp\left(-j\beta_2|\vec{r}\,'-\vec{r}|\right)/|\vec{r}\,'-\vec{r}| \tag{25}$$

where $\beta_2=\omega\sqrt{\mu_2\epsilon_2}$, and substituting $\vec{P}$ and $\vec{Q}$ into (3), we have

$$\int_{V_2}\left[-j\omega\mu_2\vec{J}_2\phi_2-\vec{M}_2\times\nabla'\phi_2+\frac{\rho_2}{\epsilon_2}\nabla'\phi_2\right]dv' = \int_{S_2}\left[-j\omega\mu_2\left(\hat{n}_2\times\vec{H}_2\right)\phi_2+\left(\hat{n}_2\times\vec{E}_2\right)\times\nabla'\phi_2 + \left(\hat{n}_2\cdot\vec{E}_2\right)\nabla'\phi_2\right]ds'. \tag{26}$$

The total boundary surface S_2 for V_2 is

$$S_2=S+S_a$$

where S_a is the surface of a small sphere (or hemisphere) for excluding the singularity point $\vec{r}$. It is noted that the infinite spherical surface S_∞ is not needed because V_2 is a finite volume.

Following the same manipulation used for the case of region 1, we can obtain $\vec{E}_2(\vec{r})$ at an interior point within

V_2 as

$$\vec{E}_2(\vec{r}) = \frac{1}{4\pi}\int_{V_2}\left[-j\omega\mu_2\vec{J}_2\phi_2 - \vec{M}_2\times\nabla'\phi_2 + \frac{\rho_2}{\epsilon_2}\nabla'\phi_2\right]dv' - \frac{1}{4\pi}\int_S\left[-j\omega\mu_2(\hat{n}_2\times\vec{H}_2)\phi_2 + (\hat{n}_2\times\vec{E}_2)\times\nabla'\phi_2 + (\hat{n}_2\cdot\vec{E}_2)\nabla'\phi_2\right]ds'.$$

Using the definitions of the equivalent surface currents $(\vec{J}_s, \vec{M}_s)$ given in (10) and (11), we can rewrite

$$\vec{E}_2(\vec{r}) = \frac{1}{4\pi}\int_{V_2}\left[-j\omega\mu_2\vec{J}_2\phi_2 - \vec{M}_2\times\nabla'\phi_2 + \frac{\rho_2}{\epsilon_2}\nabla'\phi_2\right]dv' + \frac{1}{4\pi}\int_S\left[-j\omega\mu_2(-\vec{J}_s)\phi_2 - (-\vec{M}_s)\times\nabla'\phi_2 + \frac{(-\rho_s)}{\epsilon_2}\nabla'\phi_2\right]ds'$$

$$(\vec{r}\text{ is an interior point within } V_2). \quad (27)$$

Notice that the equivalent surface currents which can maintain the correct $\vec{E}$ field inside V_2 are $(-\vec{J}_s, -\vec{M}_s)$, which flow in opposite directions on S compared with the case of region 1. Equation (27) implies that when the source currents $(\vec{J}_1, \vec{M}_1)$ in V_1 are removed and the medium of region 1 is replaced by that of region 2 (to make the whole space homogeneous), the correct value of the electric field at an interior point $\vec{r}$ inside V_2 can be calculated from the source currents $(\vec{J}_2, \vec{M}_2)$ in V_2 and the negative equivalent surface currents $(-\vec{J}_s, -\vec{M}_s)$ on S.

Similarly, the electric field at a field point on S can be expressed as

$$\vec{E}_2(\vec{r}) = \frac{1}{2\pi}\int_{V_2}\left[-j\omega\mu_2\vec{J}_2\phi_2 - \vec{M}_2\times\nabla'\phi_2 + \frac{\rho_2}{\epsilon_2}\nabla'\phi_2\right]dv' + \frac{1}{2\pi}\int_S\left[-j\omega\mu_2(-\vec{J}_s)\phi_2 - (-\vec{M}_s)\times\nabla'\phi_2 + \frac{(-\rho_s)}{\epsilon_2}\nabla'\phi_2\right]ds'$$

$$(\vec{r}\text{ is on } s). \quad (28)$$

The electric field at a point outside V_2 can be shown to be zero,

$$\vec{E}_1(\vec{r}) = 0\ (\vec{r}\text{ is outside } V_2) \quad (29)$$

when it is maintained by $(\vec{J}_2, \vec{M}_2)$ in V_2 and $(-\vec{J}_s, -\vec{M}_s)$ on S after $(\vec{J}_1, \vec{M}_1)$ in V_1 are removed and the whole space is filled with the medium of region 2.

Results for the $\vec{H}$ field in region 2 are similar to those given by (21) to (23) and are omitted here for brevity.

Fig. 3 depicts the results obtained above for region 2. Again, these results are consistent with the equivalence principle.

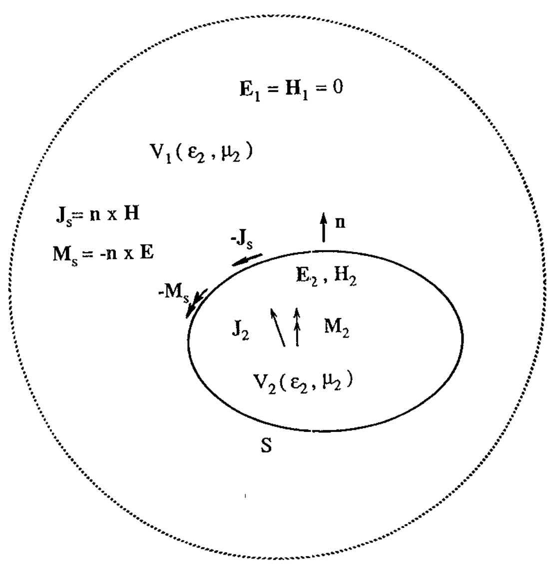

Fig. 3. When the source currents (J_1, M_1) in V_1 are removed and the medium of region 1 is replaced with that of region 2, the source currents (J_2, M_2) in V_2 and the negative equivalent surface currents $(-J_s, -M_s)$ on S will maintain the correct EM field (E_2, H_2) in V_2 and zero EM field $(E_1 = H_1 = 0)$ in V_1.

III. Applications

Mathematical formulations of the equivalence principle derived in the preceding section may have many applications. An example is given here. A finite homogeneous body of arbitrary shape with complex permittivity and permeability of (ϵ, μ) located in space is exposed to an impressed EM field with an electric field $\vec{E}^{\text{in}}$ and a magnetic field $\vec{H}^{\text{in}}$. We aim to determine the induced EM field inside the body. To solve this problem, we will first derive two integral equations for the equivalent surface currents, $\vec{J}_s = \hat{n}\times\vec{H}$ and $\vec{M}_s = -\hat{n}\times\vec{E}$, on the body surface in terms of $\vec{E}^{\text{in}}$ and $\vec{H}^{\text{in}}$. After solving for $\vec{J}_s$ and $\vec{M}_s$, the induced EM field inside the body can be easily calculated.

Let us use the same geometry as that in Fig. 1. The body is represented by region 2 with $\vec{J}_2$ and $\vec{M}_2$ removed. Region 1 represents free space, and $\vec{J}_1$ and $\vec{M}_1$ the source currents for the impressed EM field.

The $\vec{E}$ field at a point $\vec{r}$ on the body surface S in region 1 side is given by (16) as

$$\vec{E}_1(\vec{r}) = \frac{1}{2\pi}\int_{V_1}\left[-j\omega\mu_1\vec{J}_1\phi_1 - \vec{M}_1\times\nabla'\phi_1 + \frac{\rho_1}{\epsilon_1}\nabla'\phi_1\right]dv' + \frac{1}{2\pi}\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1 - \vec{M}_s\times\nabla'\phi_1 + \frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds'.$$

The volume integral of the above equation can be easily identified as twice the impressed electric field at the body surface, or it is equal to $2\vec{E}^{\text{in}}(\vec{r})$. Thus,

$$\vec{E}_1(\vec{r}) = 2\vec{E}^{\text{in}}(\vec{r}) + \frac{1}{2\pi}\int_S\left[-j\omega\mu_1\vec{J}_s\phi_1 - \vec{M}_s\times\nabla'\phi_1 + \frac{\rho_s}{\epsilon_1}\nabla'\phi_1\right]ds'. \quad (30)$$

The $\vec{E}$ field at the same point $\vec{r}$ on S but in region 2 side is given by (28) as

$$\vec{E}_2(\vec{r}) = \frac{1}{2\pi}\int_S\left[-j\omega\mu_2(-\vec{J}_s)\phi_2 - (-\vec{M}_s)\times\nabla'\phi_2 + \frac{(-\rho_s)}{\epsilon_2}\nabla'\phi_2\right]ds' \quad (31)$$

because $\vec{J}_2$ and $\vec{M}_2$ have been removed.

Since the tangential component of the $\vec{E}$ field is continuous across S, or $\hat{n}\times\vec{E}_1 = \hat{n}\times\vec{E}_2$, we can obtain from (30) and (31) an integral equation as

$$\hat{n}\times\int_S\left[j\omega\vec{J}_s(\mu_2\phi_2+\mu_1\phi_1) + \vec{M}_s\times\nabla'(\phi_2+\phi_1) - \rho_s\nabla'\left(\frac{\phi_2}{\epsilon_2}+\frac{\phi_1}{\epsilon_1}\right)\right]ds' = 4\pi\hat{n}\times\vec{E}^{\text{in}}(\vec{r}). \quad (32)$$

Similarly, from the continuity of the tangential component of the $\vec{H}$ field across S, or $\hat{n}\times\vec{H}_1 = \hat{n}\times\vec{H}_2$, we can derive another integral equation as

$$\hat{n}\times\int_S\left[j\omega\vec{M}_s(\epsilon_2\phi_2+\epsilon_1\phi_1) - \vec{J}_s\times\nabla'(\phi_2+\phi_1) - \rho_{ms}\nabla'\left(\frac{\phi_2}{\mu_2}+\frac{\phi_1}{\mu_1}\right)\right]ds' = 4\pi\hat{n}\times\vec{H}^{\text{in}}(\vec{r}). \quad (33)$$

These two integral equations can be numerically solved to determine $\vec{J}_s$ and $\vec{M}_s$ by using the method of moments and vector basis functions with triangular patch modeling [3]. After $\vec{J}_s$ and $\vec{M}_s$ are determined, the $\vec{E}$ field inside the body can be easily computed by using (27).

As a numerical example, the equivalent electric and magnetic surface currents, $\vec{J}_s$ and $\vec{M}_s$, induced by a plane EM wave on the surface of a dielectric sphere have been computed based on (32) and (33), and the results are shown in Figs. 4 and 5. The electrical size of the sphere is $\beta_1 a = 1$, where β_1 is the free-space propagation constant and a is the radius of the sphere. The permittivity of the sphere is $\epsilon_2 = 4\epsilon_0$ and the permeability is $\mu_2 = \mu_0$. The plane EM wave is incident upon the sphere from the direction of $\theta = \pi$. The induced electric and magnetic currents along a circumferential arc on $\phi = 0$ are plotted as functions of θ in Figs. 4 and 5. Along the arc, there are two components of the electric surface current, $J_{s\phi}$ and $J_{s\phi}$, and two components of magnetic surface current, $M_{s\theta}$ and $M_{s\phi}$. The values of J_s are shown normalized to the incident magnetic field H^{in} and those of M_s are normalized by the incident electric field E^{in}.

To verify the accuracy of the numerical results, they are compared with the exact solutions of Mie series. The numerical results are indicated by small triangles and the exact solutions are plotted in solid lines in Figs. 4 and 5. It is observed that very accurate numerical results can be obtained using the present method, which is based on integral equations for the induced equivalent surface currents.

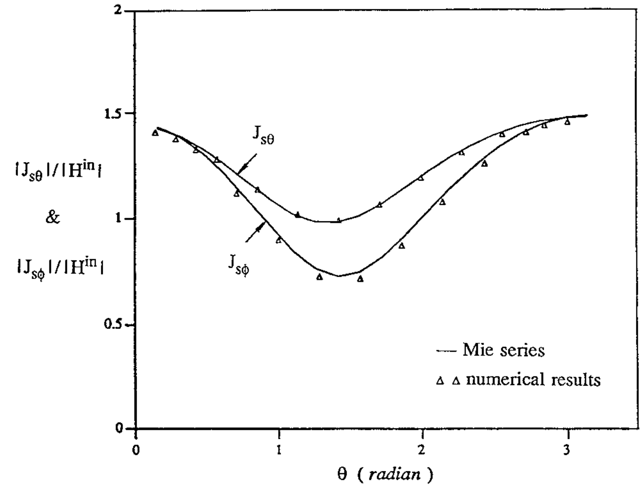

Fig. 4. Equivalent electric surface currents induced by a plane EM wave on the surface of a dielectric sphere with $\beta_1 a = 1$, $\epsilon = 4\epsilon_0$, and $\mu = \mu_0$. The plane EM wave is incident upon the sphere from the direction of $\theta = \pi$ and the surface currents are on a circumferential arc of $\phi = 0$.

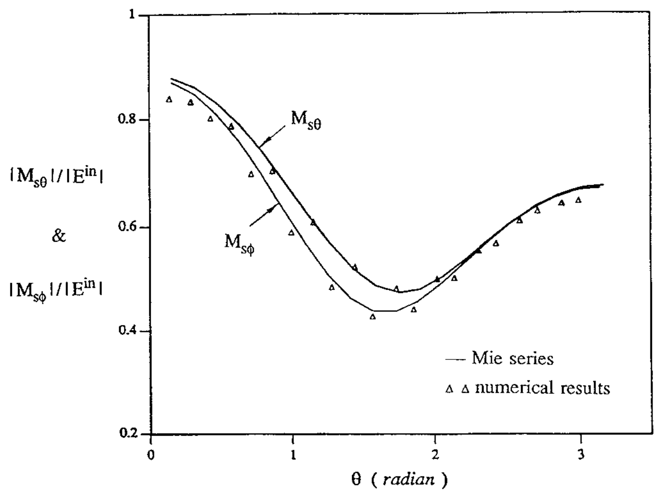

Fig. 5. Equivalent magnetic surface currents induced by a plane EM wave on the surface of a dielectric sphere with $\beta_1 a = 1$, $\epsilon = 4\epsilon_0$, and $\mu = \mu_0$. The plane EM wave is incident upon the sphere from the direction of $\theta = \pi$ and the surface currents are on a circumferential arc of $\phi = 0$.

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